CIRCUIT MATHEMATICS

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1 Outline of Unit

Detail	Duration	Comment
Three phase systems, per unit notation, matrix algebra, power invariance, transforms for 0,1,2 and dq0 Basics of travelling waves	2 unit	Be able to calculate three phase power, understand rational of transforms, calculate fault levels inc. earth fault. Appreciate the reason for the dq0 transform

2 Preliminaries

2.1 RMS

RMS stands for "root mean square".

Definition of RMS

Let waveform be $Va(t) := V \cdot \sin(\omega \cdot t)$ where $V := \sqrt{2}$

the periodic time is :- $\tau := \frac{2 \cdot \pi}{\omega}$

definition of "rms" $Vrms := \sqrt{\frac{1}{\tau} \cdot \int_0^{\tau} (Va(t))^2 \, dt}$

evaluate $Vrms = 1$

ie Vrms = Vcrest / sqrt(2)

This only applies to a purely sinusoidal waveform.

2.2 RSS

When harmonic frequencies are present the value to be computed is the RSS or "root sum square" which is defined as

Definition of RSS

The rss value is the square root of the sum of the (rms) value of each frequency squared

$$Vrss := \sqrt{\sum_{k=1}^{nmax} (Vrms_k)^2}$$

example

$Vrms_1 := 1.00$ $\quad Vrms_3 := 0.025$

$Vrms_5 := 0.10$ $\quad Vrms_7 := 0.05$

$Vrss = 1.007$

2.3 Power (energy)

This is the work done as is defined as

Power

current $\quad i(t) := I\cdot(\cos(\alpha) + \sin(\alpha)\cdot j)\cdot\sin(\omega\cdot t)$

phase angle (radians) $\quad \alpha = 0$ $\quad$ ie zero degree lagging

Periodic time $\quad T := \dfrac{\omega}{2\cdot\pi}$

evaluation $\quad P := \dfrac{1}{T}\cdot\int_0^T Va(t)\cdot i(t)\,dt$

result $\quad P = 1$

2.4 Power Factor

There are many definitions ranging from simple single frequency circuits to non-linear three phase systems.

In the simple case, power factor is the cosine of the angle between the phase voltage and the phase current. Note that I*cos(angle) is equal to the in-phase (power) component of the current thus for a single phase circuit

Power = Vphase * Iline * Power factor.

2.5 kVAr

This is sometimes referred to as “reactive power”.

$$Q := Vrms\cdot Irms\cdot\sin(angle)$$

Since the energy associated with Q oscillates between the inductance and capacitance of the system with no net work done, the use of the term "power" is inaccurate. Power engineers usually use the term "VAr" Volt Ampere Reactive.

2.6 Complex Volt-amperes

The complex Volt-amperes are defined as

$$S = V^t * I^{conj}$$

In single phase systems this reduces to

S = product Volt-Amperes = mod(V) * mod(I)* arg(-1*(argV – argI))

= $P + j.Q$

where P = power (kW) and
Q = VArs

NB :-
the definition of VAr is important "positive VAr are consumed by an inductive load and generated by a capacitive load" From this definition follows `
(1) a capacitive load consumes negative VAr
(2) S is calculated as voltage times the conjugate of current

NB :-
for matrix use both V & I are single column vectors so the transpose of V must be taken.

$$S^{\wedge 2} = P^{\wedge 2} + Q^{\wedge 2}$$

When transform techniques are used to facilitate calculation, the basis for power calculation may change.

Let $V_a = Z_a * I_a$ and $V_f = Z_f * I_f$

were "a" represents the normal a,b,c three phase system and "f" represents that system transformed to some other reference frame such that

$V_a = C_{af} * V_f$ and $I_a = C_{af} * I_f$

Thus $Sa = (C_{af} * V_f)^t * (C_{af} * I_f)^{conj}$

$$= V_f^{\,t} * (C_{af}^{\,t} * C_{af}^{\,conj}) * I_f^{\,conj}$$

If the product $C_{af}^{\ t} * C_{af}^{\ conj}$ equals the unit matrix, then the transform is said to be "power invariant".

2.7 Per Unit Notation

Many engineering steady state calculations can be simplified by the use of per unit notation. The idea is that all Voltage, Current and impedance parameters can be expressed to a common kVA base value. The use of per unit impedance is particularly useful as most motors and transformers have a near constant per unit impedance irrespective of the rating.

Percentage values are sometimes seen; this is of historic use in calculations but still quoted for parameters and you can easily get tripped up by not multiplying or dividing by 100 !

Procedure
(1) select a kVA or MVA "Base"
(2) using the nominal voltage for that system convert all parameters

Vpu = Vactual / Vnominal
Irated = Base / (sqrt(3) * Vl)
Z pu = Z ohms * Base / Vline^2.

Example :
The impedance of a 350 kVA 11kV/415V transformer is 4.5%, the X:R ratio is 4.0 and supplies rated load at 0.707 power factor lagging. Calculate the secondary voltage on the assumption that there is no voltage regulation in the supply.

Select a base

$$\text{Base} := 350 \cdot 10^3$$

write the impedance in complex notation

$$\text{ztrf} := 0.045 \left(\frac{1 + 4 \cdot j}{\sqrt{1 + 16}} \right)$$

secondary power factor

$$\text{pf} := 0.707$$

$$\alpha := \text{acos}(\text{pf}) \qquad \alpha \cdot \frac{180}{\pi} = 45.009 \quad \text{degrees}$$

secondary current

$$\text{I2} := 1.0 \cdot (\cos(\alpha) - \sin(\alpha) \cdot j)$$

$$\text{I2} = 0.707 - 0.707i$$

calculate secondary voltage

$$\text{V2} := 1.00 - \text{ztrf} \cdot \text{I2}$$

$$\text{V2} = 0.961 - 0.023i \quad \text{per unit}$$

$$|\text{V2}| = 0.962$$

Select a base $Base := 632 \cdot 10^3$

Own base for rtf1 $ownbase := 350 \cdot 10^3$

write the impedance in complex notation $ztrf := 0.045 \frac{Base}{ownbase} \cdot \left(\frac{1 + 4 \cdot j}{\sqrt{1 + 16}} \right)$

impedance for 2nd trf $ztrf2 := 0.065 \frac{Base}{1050 \cdot 10^3} \cdot \left(\frac{1 + 4 \cdot j}{\sqrt{1 + 16}} \right)$

trf impedances are $ztrf = 0.02 + 0.079i$

$ztrf2 = 9.489 \times 10^{-3} + 0.038i$

secondary power factor $pf := 0.707$

$\alpha := acos(pf)$

secondary current $I2 := 1.0 \cdot \frac{Base}{ownbase} \cdot (\cos(\alpha) - \sin(\alpha) \cdot j)$

$I2 = 1.277 - 1.277i$

calculate secondary voltage $V2 := 1.00 - (ztrf + ztrf2) \cdot I2$

$V2 = 0.814 - 0.112i$ per unit

$|V2| = 0.821$

Changing the Base in mixed calculations;
Repeat the above calculation using a base of 632 kVA and cascade a second transformer of 6.5% 1050 kVA

3 Three phase systems

The origin of polyphase systems can be found in the economic incentives of

Reduction in voltage drops allows smaller cables to be used in any given location,

Reduction in return circuit current reduces the cross-section and hence cost of the neutral,

Robust rotating machines (see later Notes)

In the same way that opposing mechanical forces are used to stabilise structures, so opposing electrical "forces" can be used to cancel each other out. A pole balanced on it end is stabilised in the vertical plane by the use of at least 3 forces arranged such that :-

$$\mathrm{Fnet} := \sum_{k=1}^{\max} F_k$$

Question 1 : Why is a minimum of three forces required to stabilise the pole.

Alternating current ("ac") is a sinusoidal ebb and flow of electrons; thus at any instant in time it can be represented by a phasor with magnitude and angular displacement from the reference axis, i.e. $\mathbf{F_k}$ = f * arg(alpha$_f$). If we select two more phasors equal in magnitude to "f" and make their phase displacement relative to be $\mathbf{F_k}$ to be +/- 120 degrees then the three phasors will form a closed triangle.

This can be generalised to say that a balanced polyphase system of "k" phases requires that the phase displacement between each phasor is 360/k

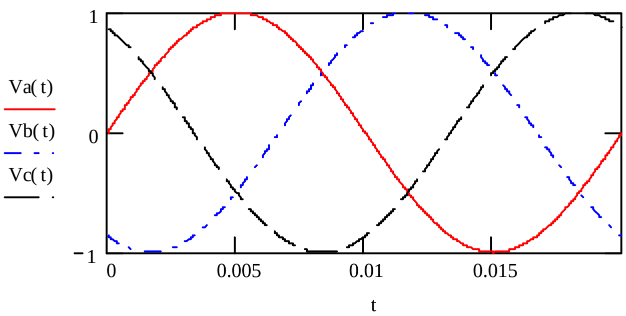
Historically the 3 phase system was seen as a good compromise. There is however no analytic reason why a 13 phase system could not be used.

$$\omega := 100 \cdot \pi$$

$$Va(t) := V \cdot \sin(\omega \cdot t)$$

$$Vb(t) := V \cdot \sin\left(\omega \cdot t - \frac{2 \cdot \pi}{3}\right)$$

$$Vc(t) := V \cdot \sin\left(\omega \cdot t - \frac{4 \cdot \pi}{3}\right)$$



The normal domestic and industrial voltages are derived from a 415 V three phase system.

Vphase-neutral = Vline-line / sqrt(3)
= 415 / 1.73
= 240.

Three phase power :-

Pabc = Pa + Pb +Pc
= 3 * Vphase * Iline

= root(3) * Vline * Iline* cos(angle)

Qabc = root(3) * Vline * Iline * sin(angle)

Note the assumption that the voltages and currents are assumed to be perfectly balanced (magnitudes equal and exactly 120 degrees between phasors).

3.1 Unbalanced loads, neutral currents and three wire systems

If the impedance of the supply and cables is ignored

3 phase operator $h := \frac{-1}{2} + \frac{\sqrt{3}}{2} \cdot j$

phase "a"	$a := 15 \cdot \frac{\pi}{180}$	$Za := 10 \cdot (\cos(a) + \sin(a) \cdot j)$	$Va := 240$
phase "b"	$b := 7 \cdot \frac{\pi}{180}$	$Zb := 12 \cdot (\cos(b) + \sin(b) \cdot j)$	$Vb := Va \cdot h^2$
phase "c"	$c := 32 \cdot \frac{\pi}{180}$	$Zc := 4 \cdot (\cos(c) + \sin(c) \cdot j)$	$Vc := Va \cdot h$

If there is no influence from the supply or cables, and if the neutral is solidly grounded at the transformer

$Ia := \frac{Va}{Za}$ $\quad Ib := \frac{Vb}{Zb}$ $\quad Ic := \frac{Vc}{Zc}$ $\quad In := -(Ia + Ib + Ic)$

	phasor	modulus
phase "a"	$Ia = 23.182 - 6.212i$	$\lvert Ia \rvert = 24$
phase "b"	$Ib = -12.036 - 15.973i$	$\lvert Ib \rvert = 20$
phase "c"	$Ic = 2.094 + 59.963i$	$\lvert Ic \rvert = 60$
neutral	$In = -13.24 - 37.779i$	$\lvert In \rvert = 40.032$

Next repeat the calculation with no neutral cable. What are the line currents and the neutral voltage. Matrix solution makes life a lot easier.

No neutral cable; use Nodal admittance matrix

there are 4 nodes plus the reference node (earth),
required to calculate Vn and Iabc

$$ya := Za^{-1} \qquad yb := Zb^{-1} \qquad yc := Zc^{-1}$$

write down the nodal admitance matrix by inspection

$$Y := \begin{pmatrix} ya & 0 & 0 & -ya \\ 0 & yb & 0 & -yb \\ 0 & 0 & yc & -yc \\ -ya & -yb & -yc & ya + yb + yc \end{pmatrix}$$

the solution is:- $I := Y \cdot V$

where
all voltages are w.r.t. earth
In = 0
but the neutral voltage is not known

solve the neutral voltage first

$$Vn := \frac{ya \cdot Va + yb \cdot Vb + yc \cdot Vc}{ya + yb + yc} \qquad Vn = -6.53 + 93.731i$$

hence the nodal voltage matrix is

$$Vabcn := \begin{pmatrix} Va \\ Vb \\ Vc \\ Vn \end{pmatrix}$$

then solve for currents $Iabcn := Y \cdot Vabcn$

to give

$$Ia = 21.387 - 15.434i \qquad |Ia| = 26.375$$

$$Ib = -12.448 - 23.792i \qquad |Ib| = 26.851$$

$$Ic = -8.939 + 39.226i \qquad |Ic| = 40.232$$

$$In = 3.768i \times 10^{-15} \qquad |In| = 3.841 \times 10^{-15}$$

3.2 Star – Delta Transforms

The easy way to remember the equation is

D = Straddle / Sum

For Y – to –D use admittance values, for D – to – Y use impedances.

3.3 Assignments

Prove the Star-Delta formula using the nodal admittance matrix.

2. Solve the following :

A three phase 4 wire supply has the following loads connected

1. phase a-n 15kW electric heating load

2. phase b-n 25kVAr capacitor

3. phase c-n 12kVa 0.72 lagging power factor load

4. three phase delta connected balanced load of 10kVA, 0.5 power factor lagging

All load values are described on the basis of a perfect 415/240V supply.

Calculate the individual phase currents phase to neutral and line to line voltages for the cases of (1) a healthy 3 phase supply & (2) with the neutral broken.

For the second condition calculate the phase to earth voltages as well as the phase to neutral (star) point voltages.

Sketch phasor diagrams to explain and validate the answers

3. Explain how real power (kW) may be measured in an unbalanced 3 phase system.

4 Transform Techniques

Polyphase circuits present special problems in terms of solving unbalanced conditions and incorporating mutual coupling. Matrix methods can be used to eliminate these problems.

The basic concept is to find a mathematical method to diagonalise the characteristic equations being worked with.

4.1 Symmetrical components

The impedance matrix for a triangular construction transmission line above an infinitely conducting earth is characterised by equal terms on the leading diagonal and equal terms off-diagonal. The same structure applies to transformers and passive equipments.

Symmetrical Components

define the "h" operator $h := \frac{-1}{2} + \frac{\sqrt{3}}{2} \cdot j$

and the "H" matrix $H := \begin{pmatrix} 1 & 1 & 1 \\ 1 & h^2 & h^1 \\ 1 & h^1 & h^2 \end{pmatrix}$

in rms terms the three pahse voltages can be written $Vabc := \begin{pmatrix} 1 \\ h^2 \\ h^1 \end{pmatrix}$

apply H to these voltages $V012 := H^{-1} \cdot Vabc$ etc

giving $V012 = \blacksquare$

Applying this to the symmetrical impedance matrix gives :-

Applying Ohms Law gives the impedance trasnform

$$Vabc := Zabc \cdot Iabc$$

$$Vabc := Zabc \cdot H^{1} \cdot I012$$

$$V012 := H^{-1} \cdot Zabc \cdot H^{1} \cdot I012$$

$$Z012 := H^{-1} \cdot Zabc \cdot H$$

balanced impedance matrix

$$Zabc := \begin{pmatrix} zs & m & m \\ m & zs & m \\ m & m & zs \end{pmatrix}$$

put in some values

$$zs = 6 \qquad m = 1$$

transformed value

$$Z012 = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Calculation of power in the 0,1,2 frame is not “invariant” viz.

$$H^{T} \cdot \overline{H} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Thus the symmetrical component transform is not power invariant.

Question : what modification is required to make the transform power invariant and what effect does this have on the concept of using this transform.

4.2 Fault analysis with symmetrical components

Write down the unique conditions ***at the point of fault***, i.e. for a single phase fault

Va=0, Ib=Ic=0

From these conditions it can be shown that

I0 = I1 = I2, V0 + V1 + V2 = 0

from which it follows that

I1 = V1 / (Z0 + Z1 + Z2)

this corresponds to a series connection of the three “sequence component networks”

~~Assignment~~ : derive the point of fault connection of the sequence networks for both phase to phase clear of earth and phase to phase including earth.

4.3 D,Q,0 transform

Rotating machines present a special case because the mutual coupling between phases depends on the direction of rotation relative to the physical position of the various coils, i.e.

A number of transforms have been developed over the years. The most enduring being that associated with Park. Several slightly different versions of this transform exist. The one we present and prefer is that

DQ0 Transform

define phase self inductances $Laa := Ls + Lm\cdot\cos(2\cdot\varphi)$

$$Lbb := Ls + Lm\cdot\cos\left[2\cdot\left(\varphi - 2\cdot\frac{\pi}{3}\right)\right]$$

$$Lcc := Ls + Lm\cdot\cos\left[2\cdot\left(\varphi + 2\cdot\frac{\pi}{3}\right)\right]$$

define stator mutuals

$$Lab := -M - Lm\cdot\cos\left[2\left(\varphi + \frac{\pi}{6}\right)\right] \qquad Lba := Lab$$

$$Lbc := -M - Lm\cdot\cos\left[2\left(\varphi - \frac{\pi}{2}\right)\right] \qquad Lcb := Lbc$$

$$Lca := -M - Lm\cdot\cos\left[2\left(\varphi + \frac{5\cdot\pi}{6}\right)\right] \qquad Lac := Lca$$

machine impedance matrix

$$Labc := \begin{pmatrix} Laa & Lab & Lac \\ Lba & Lbb & Lbc \\ Lca & Lcb & Lcc \end{pmatrix}$$

Let !!!

$$P := \sqrt{\frac{2}{3}}\cdot\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \cos(\varphi) & \cos\left(\varphi - 2\cdot\frac{\pi}{3}\right) & \cos\left(\varphi + 2\cdot\frac{\pi}{3}\right) \\ \sin(\varphi) & \sin\left(\varphi - 2\cdot\frac{\pi}{3}\right) & \sin\left(\varphi + 2\cdot\frac{\pi}{3}\right) \end{pmatrix}$$

which preserves “power invariance”.

Inserting some typical values and applyingthe impedance transform gi

$$Lm := 0.2 \quad M := 1.2 \cdot Lm \quad Ls := 3. \cdot M \quad \varphi := 20 \cdot \frac{\pi}{180}$$

$$P \cdot Labc \cdot P^{-1} = \begin{pmatrix} 0.24 & 0 & 0 \\ 0 & 1.26 & 0 \\ 0 & 0 & 0.66 \end{pmatrix}$$

~~Assignment~~ : complete the symbolic solution for L_{0dq} above and solve for L0, Ld and Lq

4.4 Use of DQ axis parameters in practice

when using dq parameters for synchronous machines there is an underlying assumption that the speed of the machine is constant. This comes from the fact that we solve an equation of the form

V = +- Sum(r*I) +- Sum(rate of change of flux linkages per phase)

The latter term is not the same as L*di / dt because the inductance varies as a position of rotor angle (see above) so the solution is actually

V = +- Sum(r*I) +- Sum(L*di / dt + I*dL / dt)

When working with dq axis models it has been the norm to ignore the terms due to rate of change of speed on the grounds that this is usually very small however with

5 Travelling Wave Theory

5.1 Basic Concepts

Any conductor system (“transmission line”) exhibits series resistance and inductance as well as shunt capacitance. Any transmission line can be thought of as comprising a large number of elementary R,L,C section. When a step of voltage is applied to the first section, the capacitor “slowly” charges and thus the voltage applied to the second section comprises a finite rate of rise front rather than a step.

As the point of observation moves down the line, the wavefront progressively slows down and a delay is observed between the initially applied step and the voltage at the point of observation.

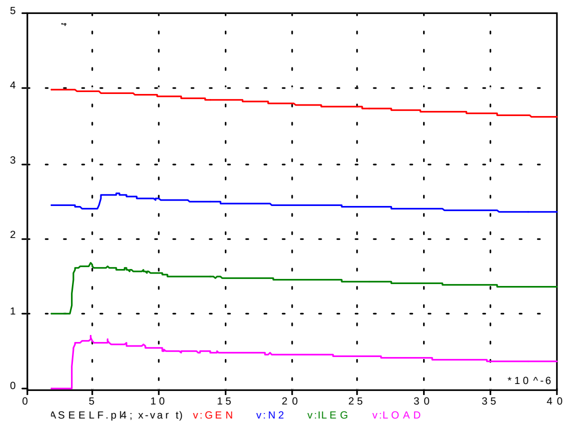
Thus the voltage front “travels” down the line.

At every point on the on the line the voltage and current are related by the surge impedance (= sqrt(L / C)) of the line.

Now consider the effect of reaching an open circuit – no current can flow thus a negative reflection must be sent back down the line and a positive voltage reflection also accompanies it. At a short circuit the converse occurs positive current and negative voltage reflection. In the case of the terminating impedance being equal to that of the line no reflections are needed. Reflection co-efficients are calculated as :-

$$Rv = (Zt - Zs) \;/\; (Zt + Zs)$$

These reflections occur in every distributed parameter system be it a water pipe, a mechanical beam or a copper track on a printed circuit board. Indeed the extent of reflections and distortion of high frequency signals in printed board trackwork determines the quality and reliability of the PCB.



Calculated voltage profile along 1km of electric fence

Basic rules

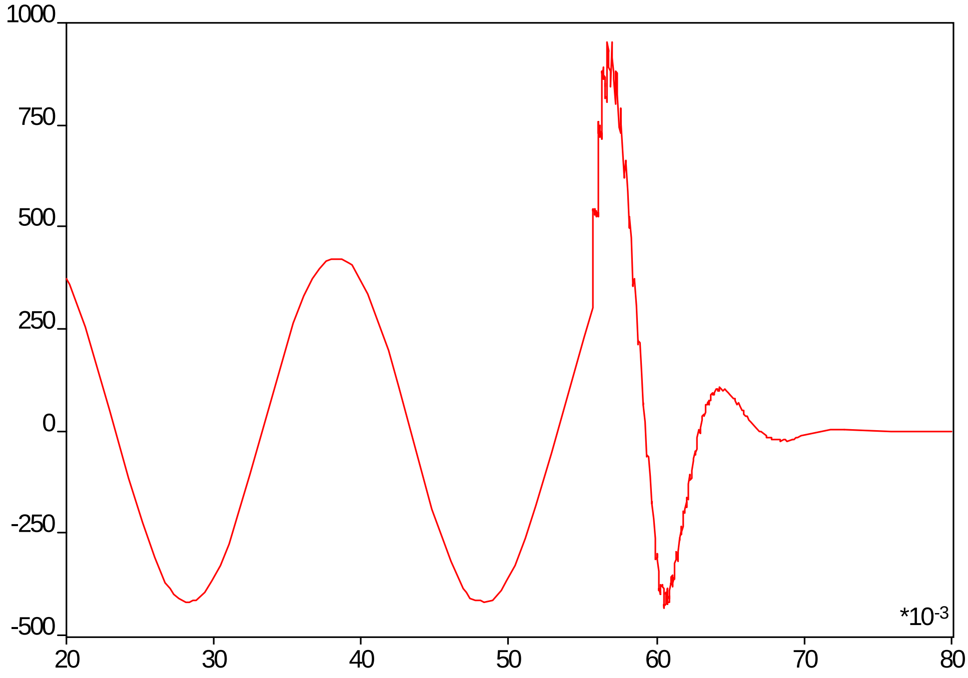
Open circuits double the voltage and eliminates the current

Short circuits double the current and eliminates the voltage.

Velocity of propagation = a bit less than 3 * 10^8 m / sec (velocity of light) for an overhead line.

Surge impedance = 200 – 400 ohms for over-head line;
= 20 - 30 ohms for HV u / g cable;
= 5 – 20 ohms for LV cable installations

Example of energisation transients on 11kV line



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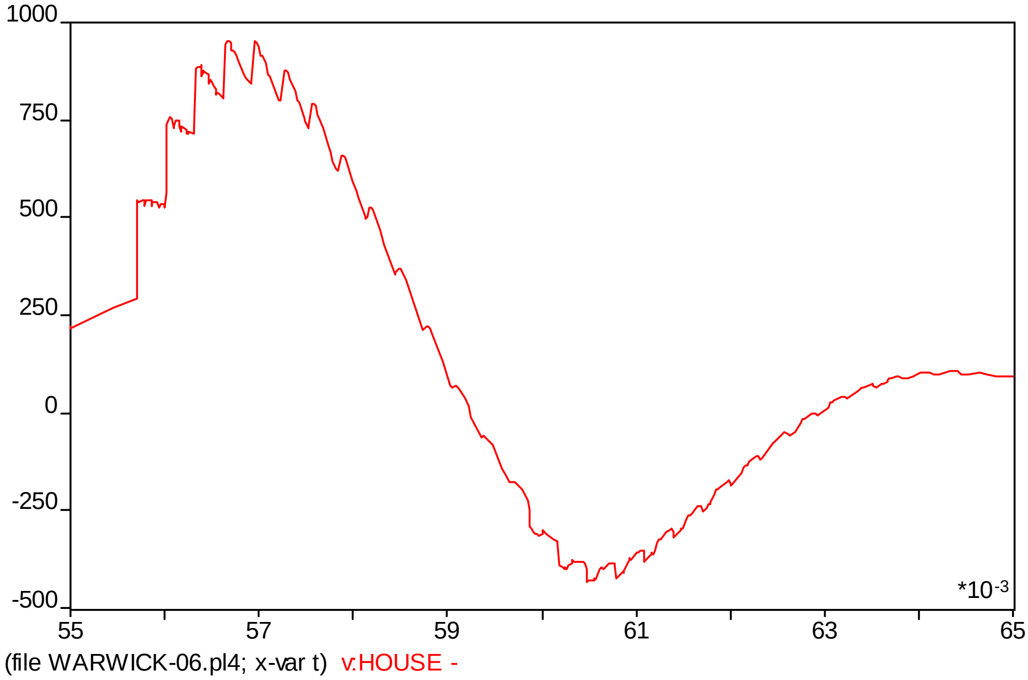
and when the detail of the transient is looked at we find :-



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Next comes the PowerPoint presentation ANU-PE01

6 Reference List

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